MATHEMATICS

ON THE COMPLETENESS OF THE QUOTIENT ALGEBRAS OF A COMPLETE BOOLEAN ALGEBRA I

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Introduction

Let A be a complete Boolean algebra and I a principal ideal of A, then it is well-known that the quotient algebra A/I is complete. Moreover the natural homomorphic mapping $A \to A/I$ is complete, which means that it preserves infinite sums (and therefore infinite products). Conversely, if A/I is complete and the homomorphic mapping $A \to A/I$ is complete, then I is a principal ideal. If I is an ideal which is not principal then A/I may or may not be complete and it follows that if A/I is complete that the homomorphic mapping $A \to A/I$ is not complete.

Accordingly, the quotient algebras A/I of a complete Boolean algebra A can be divided into three classes: (i) A/I is complete and the homomorphic mapping $A \to A/I$ is complete, (ii) A/I is complete and the homomorphic mapping $A \to A/I$ is not complete, (iii) A/I is not complete.

It is clear that if A is finite that every quotient algebra of A belongs to the class (i) and that in general for any complete Boolean algebra A the quotient algebras of the class (i) are characterised by the principal ideals of A. The main purpose of the present paper, which is the first of a projected series, is to prove that every infinite complete Boolean algebra also has quotient algebras belonging to the classes (ii) and (iii). In subsequent papers we shall discuss the problem of the characterisation of the quotient algebras of a given complete Boolean algebra A, relative to the above classification. Moreover we shall extend our results to the case of α-complete Boolean algebras (α a cardinal number).

§ 1.

In this section we shall recall some definitions and properties.

A always stands for a Boolean algebra and small characters denote elements of A. The sum (join) of two elements x and y of A is denoted by $x+y$ and the product (meet) by xy. The zero and unit element of A are denoted by 0 and 1 respectively. The complement of an element x is denoted by $\bar{x}$ or x^-. A is called *complete* if the sum (and therefore the product) of any set of elements $\{x_\gamma, \gamma \in \Gamma\}$ exists in A. If $\{x_\gamma, \gamma \in \Gamma\}$ is a set of elements of a complete Boolean algebra, then we denote the sum of this set by $\sum_{\gamma\in\Gamma} x_\gamma$ and the product by $\prod_{\gamma\in\Gamma} x_\gamma$. If X is a subset of A, then we also denote the sum and the product of the elements belonging to X by $\sum X$ and $\prod X$ respectively. A subset of a Boolean algebra is called *disjoint* if $xy=0$ for every pair of distinct elements x and y of that subset. If A is complete then for every set of elements $\{x_\gamma, \gamma \in \Gamma\}$ we have $(\sum_{\gamma\in\Gamma} x_\gamma)^- = \prod_{\gamma\in\Gamma} \bar{x}_\gamma$. An element $x \in A$, $x \neq 0$ is called an *atom* if $y \leqslant x$ implies that either $y=0$ or $y=x$. A *subalgebra* of a Boolean algebra A is a subset of A closed under the operations sum, product and complementation. A subalgebra A' of a complete Boolean algebra A is called a $\sum, \prod$-complete subalgebra of A if A' is also closed under infinite sums and infinite products. It is clear, that A' can be complete without being a $\sum, \prod$-complete subalgebra.

It is well-known that every complete Boolean algebra A satisfies the following infinite distributive laws:

$$y \sum_{\gamma\in\Gamma} x_\gamma = \sum_{\gamma\in\Gamma} y x_\gamma \tag{1}$$

$$y + \prod_{\gamma\in\Gamma} x_\gamma = \prod_{\gamma\in\Gamma} (y + x_\gamma). \tag{2}$$

It is not difficult to show that the following relation follows from (1):

$$\sum_{\gamma\in\Gamma} x_\gamma \sum_{\lambda\in\Lambda} y_\lambda = \sum_{\gamma\in\Gamma, \lambda\in\Lambda} x_\gamma y_\lambda. \tag{3}$$

An *ideal* of a Boolean algebra A is a non-void subset of A closed under (finite) sums and containing with every element x all elements $t, t \leqslant x$. A *principal ideal* is an ideal which has a greatest element a and is denoted by (a). If I is an ideal of A, then $A-I$ denotes the set $\{x, x \in A, x \notin I\}$.

If $\{I_\gamma, \gamma \in \Gamma\}$ is a set of ideals of A, then it is easy to see that the ideal generated by this set of ideals is the ideal, the elements of which can be written as $\sum_{\gamma\in\Gamma_1} x_\gamma$, where Γ_1 is any finite subset of Γ. If I is an ideal of A, then for every $x \in A$, x^* stands for the image of x in the quotient algebra A/I under the natural homomorphic mapping $A \to A/I$. It is well-known that the ideals of A are in one-one correspondence with the homomorphic mappings of A – up to isomorphic mappings – such that every ideal is the kernel of the corresponding homomorphic mapping.

An ideal I of A is a *prime ideal* if (i) $I \neq A$ (ii) $xy \in I$ implies either $x \in I$ or $y \in I$. It is well-known [1] that I is prime if and only if one of the following equivalent conditions is satisfied: (iii). For every $x \in A$, *one* of the elements x and $\bar{x}$ belong to I, (iv) A/I is the two elements Boolean algebra, (v) I is a maximal ideal. Finally we recall, that every proper ideal of a Boolean algebra can be extended to a prime ideal. [2]

The following lemmas are useful.

Lemma 1.1

For every two elements x and $y \in A$, we have $x+y=x+\bar{x}y$.

The proof is immediate.

Lemma 1.2

If I is an ideal of A, then $x^* \geqslant y^*$ if and only if $\bar{x}y \in I$. In particular $x \geqslant y$ if and only if $\bar{x}y = 0$.

Proof.

If $x^* \geqslant y^*$, then $x^*y^* = y^*$. Thus $0^* = \bar{x}^*x^*y^* = \bar{x}^*y^* = (\bar{x}y)^*$ thus $\bar{x}y \in I$. Conversely if $\bar{x}y \in I$, then $y = y(x+\bar{x}) = yx + y\bar{x}$. Thus $y^* = y^*x^*$ or $x^* \geqslant y^*$.

§ 2.

We start this section with the proof of the following known theorem [3].

Theorem 2.1

If X is a non-void subset of a complete Boolean algebra A, then there exists a disjointed subset Y of A, such that $\sum X = \sum Y$ and such that for every $y \in Y$, there exists an $x \in X$, such that $y \leqslant x$.

Proof.

We suppose the elements of X to be well-ordered: $x_1, x_2, \ldots, x_\omega, \ldots$ and we define the elements $y_1, y_2, \ldots, y_\omega, \ldots$ as follows: $y_1 = x_1$ and for every $\alpha > 1$, $y_\alpha = x_\alpha[\sum_{\xi<\alpha} x_\xi]^-$. It is clear that the set Y is disjointed. We shall show that for every $\alpha \geqslant 1$, we have $x_\alpha \leqslant \sum Y$. It is clear that this is true for $\alpha = 1$. Suppose that we know that for some $\alpha > 1$, $x_\xi \leqslant \sum Y$ for every $\xi < \alpha$, then $\sum_{\xi<\alpha} x_\xi \leqslant \sum Y$ and thus $x_\alpha \sum_{\xi<\alpha} x_\xi \leqslant \sum Y$. But also $x_\alpha\, [\sum_{\xi<\alpha} x_\xi]^- = y_\alpha \leqslant \sum Y$. thus $x_\alpha \leqslant \sum Y$. Thus it follows that $\sum X \leqslant \sum Y$. On the other hand we have that for every α, $y_\alpha \leqslant x_\alpha$ thus $\sum Y \leqslant \sum X$ and thus $\sum X = \sum Y$.

We observe that the cardinal number of the set Y need not be equal to the cardinal number of X. Take for instance the Boolean algebra $A = 2^{\aleph_0}$. Every disjointed subset of A has at most $\aleph_0$ elements, but A has continuously many elements. On the other hand we can prove the following theorem, which plays a basic rôle in the sequel.

Theorem 2.2

If X is a countably infinite subset of a complete Boolean algebra A, then there exists a countably infinite disjointed subset Y of A such that $\sum X = \sum Y$. In particular, every infinite complete Boolean algebra contains a countably infinite disjointed subset.

Proof.

We denote the set X by $\{x_i,\ i=1, 2, \ldots\}$, $x_i \neq x_j$, $i \neq j$ and let $a = \sum_{i=1}^{\infty} x_i$. We may assume that $0 < x_1 < a$. Let $x_1' = a\bar{x}_1$. Define the sets X_1 and X_1' by $X_1 = \{x_i x_1,\ i=1, 2, \ldots\}$ and $X_1' = \{x_i x_1',\ i=1, 2, \ldots\}$. Now at least one of the sets X_1 and X_1' contains an infinite number of unequal elements. Indeed if X_1 and X_1' would be finite, then for some i and j, $i \neq j$, we would have $x_i x_1 = x_j x_1$ and $x_i x_1' = x_j x_1'$ or $x_i(x_1 + x_1') = x_j(x_1 + x_1')$. However, $x_1 + x_1' = x_1 + a\bar{x}_1 = x_1 + a = a$, according to Lemma 1.1. Thus it would follow, that

$x_i a = x_j a$ and thus $x_i = x_j$. Now we shall define by induction a set of elements $y_1, y_2, \ldots$.

Let $y_1 = x_1$ in case X_1' contains an infinite number of unequal elements and let $y_1 = x_1'$ in case X_1 contains an infinite number of unequal elements (In case X_1 and X_1' contain an infinite number of unequal elements take either $y_1 = x_1$ or $y_1 = x_1'$.)

Let the set $y_2, y_3, \ldots, y_n$ already be constructed such that $y_i \neq 0$ for $i = 1, 2, \ldots, n$, $y_i y_j = 0$ for $i \neq j$ (and thus $y_i \neq y_j$ for $i \neq j$) and $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $y_i \leqslant a$ for $i = 1, 2, \ldots, n$ and finally such that if $z_n = \sum_{i=1}^{i=n} y_i$ and $z_n' = a\bar{z}_n$, the set $X_n = \{x_i z_n', i = 1, 2, \ldots\}$ contains an infinite number of unequal elements. It is clear that y_1 satisfies all these conditions. Indeed if $y_1 = x_1$, then $x_i z_1' = x_i x_1'$ and if $y_1 = x_1'$ then $x_i z_1' = x_i x_1$, Now in order to construct y_{n+1} we proceed as follows. The set $\{x_i z_n', i = 1, 2, \ldots\}$ contains an infinite number of unequal elements, say the set $\{x_{n_i} z_n', i = 1, 2, \ldots\}$.

We may assume that $0 < x_{n_1} z_n' < z_n'$. Put $p = x_{n_1} z_n'$ and $p' = \bar{p} z_n'$, then $0 < p < z_n'$ and $0 < p' < z_n'$. Now we assert that at least one of the sets $\{x_{n_i} p\}$ and $\{x_{n_i} p'\}$ contains an infinite number of unequal elements. Otherwise we would have for some i and j, $i \neq j$, that $x_{n_i} p = x_{n_j} p$ and $x_{n_i} p' = x_{n_j} p'$ and thus $x_{n_i}(p + p') = x_{n_j}(p + p')$ or $x_{n_i} z_n' = x_{n_j} z_n'$. Now let $y_{n+1} = p$ in case the set $\{x_{n_i} p'\}$ contains an infinite number of unequal elements and let $y_{n+1} = p'$ in case the set $\{x_{n_i} p\}$ contains an infinite number of unequal elements. (If either set contains an infinite number of elements take either $y_{n+1} = p$ or $y_{n+1} = p'$.) First, suppose that $y_{n+1} = p$, thus that the set $\{x_{n_i} p'\}$ contains an infinite number of unequal elements. Then $y_{n+1} \neq 0$,

$$y_{n+1} y_i \leqslant y_{n+1} z_n = p z_n \leqslant z_n' z_n = a\bar{z}_n z_n = 0 \text{ for } i = 1, 2, \ldots, n.$$

Now $y_{n+1} = p = x_{n_1} z_n' \leqslant z_n' = a\bar{z}_n \leqslant a$, thus $y_{n+1} \leqslant a$. Now putting $z_{n+1} = \sum_{i=1}^{i=n+1} y_i$ and

$$z_{n+1}' = a\bar{z}_{n+1} = a\Big(\sum_{i=1}^{i=n+1} y_i\Big)^- = a \prod_{i=1}^{i=n+1} \bar{y}_i = a\Big(\prod_{i=1}^{i=n} \bar{y}_i\Big)\bar{y}_{n+1} =$$
$$= a\Big(\sum_{i=1}^{i=n} y_i\Big)^- \bar{y}_{n+1} = a\bar{z}_n \bar{y}_{n+1} = a\bar{z}_n \bar{p} = z_n' \bar{p} = p'.$$

Thus $x_i z_{n+1}' = x_i p'$ and thus the set $\{x_i z_{n+1}'\}$ contains an infinite number of unequal elements (namely a subset of the set $\{x_{n_i} p'\}$).

Secondly, suppose that $y_{n+1} = p'$, then the set $\{x_{n_i} p\}$ contains an infinite number of unequal elements. Again $y_{n+1} \neq 0$, since $p' \neq 0$. Now

$$y_{n+1} y_i \leqslant y_{n+1} z_n = p' z_n \leqslant z_n' z_n = a\bar{z}_n z_n = 0 \text{ for } i = 1, 2, \ldots, n.$$

Furthermore $y_{n+1} = p' = \bar{p} z_n' \leqslant z_n' = a\bar{z}_n \leqslant a$, thus $y_{n+1} \leqslant a$. Again putting $z_{n+1} = \sum_{i=1}^{i=n+1} y_i$, we have

$$z_{n+1}' = a\bar{z}_{n+1} = a_n \bar{z}_n \bar{p}' = a\bar{z}_n (z_n' \bar{p})^- = a\bar{z}_n (a\bar{z}_n \bar{p})^- = a\bar{z}_n p = p,$$

since $p = x_{n_1} z_n' = x_{n_1} a\bar{z}_n \leqslant a\bar{z}_n$ thus $x_i z_{n+1}' = x_i p$ and the set $\{x_i z_{n+1}'\}$ contains an infinite number of unequal elements (namely a subset of the set $\{x_{n_i}p\}$). Thus it follows that the set of elements $y_1, y_1, \ldots, y_n, y_{n+1}$ satisfies the conditions as stated above. Therefore there exists an infinite set of elements $y_1, y_2, \ldots, y_i \neq 0$ for every i, and $y_i y_j = 0$ for $i \neq j$ (and thus $y_i \neq y_j$ for $i \neq j$) and such that $y_i \leqslant a$ for every i and thus $\sum_{i=1}^{\infty} y_i \leqslant a$. Now define the element y by $y = (\sum_{i=1}^{\infty} y_i)^- a$, then it follows that the set Y consisting of all the elements y_i and the element y, is a countably infinite disjointed subset of A such that $\sum Y = a$.

The last part of the proof is immediate. Indeed, it is always possible to write 1 as the sum of a countably infinite number of elements.

§ 3.

In this section we shall prove that every infinite complete Boolean algebra has a quotient algebra A/I, such that A/I is complete and such that the homomorphic mapping $A \to A/I$ is not complete.

Theorem 3.1

Every infinite complete Boolean algebra A has a quotient algebra A/I, such that A/I is complete and such that the homomorphic mapping $A \to A/I$ is not complete.

Proof.

We consider two cases: (i) A is atomless, (ii) A has atoms.

(i) Let I be some prime ideal (which always exists) of A, then A/I is the two elements Boolean algebra and therefore complete. In order to show that the homomorphic mapping $A \to A/I$ is not complete, we have to show that I is not principal. Suppose $I = (a)$ for some $a \in A$, then $a < 1$ and thus $\bar{a} > 0$. Suppose $x \leqslant \bar{a}$, then $\bar{x} \geqslant a$. Consider the ideal I' generated by (a) and $(\bar{x})$. Since I is maximal, I' is either (a) or A. In the first case we have $\bar{x} \leqslant a$, but also $\bar{x} \geqslant a$, thus $\bar{x} = a$. In the second case we have $a + \bar{x} = 1$, but $\bar{x} \geqslant a$, thus $\bar{x} = 1$ and thus $x = 0$. It follows that a would be an atom.

(ii) Let $\{a_\gamma, \gamma \in \Gamma\}$ be the set of atoms of A and let K be the ideal generated by the set of ideals $\{(a_\gamma), \gamma \in \Gamma\}$. We shall show that $K \neq A$. Suppose $K = A$, then we would have $1 = \sum_{i=1}^{i=n} a_{\gamma_i}, \gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$. Then, for every $x \in A$ we would have $x = x1 = x \sum_{i=1}^{i=n} a_{\gamma_i} = \sum_{i=1}^{i=n} x a_{\gamma_i}$. But either $x a_{\gamma_i} = 0$ or $x a_{\gamma_i} = a$ and thus x could be written as the sum of a (finite) number of elements a_{γ_i}, but this again would imply that A is finite. Hence $K \neq A$. Now, K can be extended to a prime ideal I and again we shall prove that I is not principal. Suppose $I = (a)$. If $x \leqslant \bar{a}$ then $\bar{x} \geqslant a$. The ideal I' generated by $(\bar{x})$ and (a) is either (a) or A. In the first case we have $\bar{x} \leqslant a$, thus $x \geqslant \bar{a}$. but also $x \leqslant \bar{a}$ thus $x = \bar{a}$. In the second case we have $\bar{x}+a=1$ but $\bar{x}\geqslant a$ thus $\bar{x}=1$, thus $x=0$. Therefore $\bar{a}$ is an atom, thus $\bar{a}\in I$, thus also $1=a+\bar{a}\in I$, thus $A=I$, contradictory to the fact that I is prime and thus $I\neq A$.

§ 4.

In this section we shall prove that every infinite complete Boolean algebra A has a quotient algebra A/I, such that A/I is not complete. First we shall prove the following theorem.

Theorem 4.1

If A is a complete Boolean algebra and I an ideal of A and $\{x_\gamma, \gamma\in\Gamma\}$ is a set of elements of A, then $\sum\limits_{\gamma\in\Gamma} x_\gamma^*$ does not exist if and only if the following condition is satisfied: For every $p\in A$, $p\leqslant x$, where $x=\sum\limits_{\gamma\in\Gamma} x_\gamma$, there exists an element $q\in A-I$, $q\leqslant\bar{p}x$ and $q_\gamma=qx_\gamma\in I$ for every $\gamma\in\Gamma$.

Proof.

First, suppose that $\sum\limits_{\gamma\in\Gamma} x_\gamma^*$ does not exist. Let $p\leqslant x=\sum\limits_{\gamma\in\Gamma} x_\gamma$ and $px_\gamma\in I$ for every $\gamma\in\Gamma$. Then $(\bar{p}x)^- x_\gamma=(p+\bar{x})x_\gamma=px_\gamma+\bar{x}x_\gamma=px_\gamma\in I$ for every $\gamma\in\Gamma$. Thus, according to Lemma 1.2, we have that $(\bar{p}x)^*\geqslant x^*$ for every $\gamma\in\Gamma$, or denoting $\bar{p}x$ by y, $y^*\geqslant x_\gamma^*$ for every $\gamma\in\Gamma$. Since $\sum\limits_{\gamma\in\Gamma} x_\gamma^*$ does not exist, there exists an element $z\in A$, such that $z^*<y^*$, $z^*\geqslant x_\gamma^*$ for every $\gamma\in\Gamma$. Since $z^*<y^*$, we have, according to Lemma 1.2, that $\bar{z}y\in A-I$. Now let for every $\gamma\in\Gamma$, $q_\gamma=\bar{z}yx_\gamma$ and let $q=\sum\limits_{\gamma\in\Gamma} q_\gamma=\bar{z}yx$. We shall show:

(1) $q\in A-I$, (2) $q_\gamma\in I$ for every $\gamma\in\Gamma$, (3) $q\leqslant\bar{p}x$, (4) $q_\gamma=qx_\gamma$ for every $\gamma\in\Gamma$.

(1) $q=\sum\limits_{\gamma\in\Gamma} q_\gamma=\sum\limits_{\gamma\in\Gamma}\bar{z}yx_\gamma=\bar{z}yx=\bar{z}\bar{p}x=\bar{z}y\in A-I$. (2) $z^*\geqslant x_\gamma^*\Rightarrow\bar{z}x_\gamma\in I\Rightarrow$ $\Rightarrow\bar{z}yx_\gamma=q_\gamma\in I$. (3) $q_\gamma=\bar{z}yx_\gamma=\bar{z}\bar{p}xx_\gamma=\bar{z}\bar{p}x_\gamma$ thus $q_\gamma\leqslant\bar{p}x_\gamma$ for every $\gamma\in\Gamma$ and thus $q\leqslant\bar{p}x$. (4) $qx_\gamma=\bar{z}yxx_\gamma=\bar{z}yx_\gamma=q_\gamma$.

(1), (2), (3) and (4) show that q satisfies the condition of the theorem.

Secondly, syppose that for some set $\{x_\gamma, \gamma\in\Gamma\}$ of elements of A the conditions of the theorem is satisfied, then we must show that $\sum\limits_{\gamma\in\Gamma} x_\gamma^*$ does not exist. Thus we must prove that for every y, such that $y^*\geqslant x_\gamma^*$ for every $\gamma\in\Gamma$, there exists an element z, such that $z^*<y^*$ and $z^*\geqslant x_\gamma^*$ for every $\gamma\in\Gamma$.

Let $p=\bar{y}x$ and let for every $\gamma\in\Gamma$, $p_\gamma=px_\gamma=\bar{y}xx_\gamma=\bar{y}x_\gamma$. Then we have that $p\leqslant x$ and $p_\gamma\in I$ for every $\gamma\in\Gamma$. Thus there exists an element q, $q\leqslant\bar{p}x$ (and thus $q\leqslant x$), such that $q_\gamma=qx_\gamma\in I$ for every $\gamma\in\Gamma$ and such that $q\in A-I$. If $z=\bar{p}\bar{q}x$, then we shall prove that

(1) $z^*\leqslant y$, (2) $z^*\geqslant x_\gamma^*$ for every $\gamma\in\Gamma$.

(1) $z=\bar{p}\bar{q}x=(\bar{y}x)^-\bar{q}x=(y+\bar{x})\bar{q}x=y\bar{q}x$, thus $z\leqslant y$ and thus $z^*\leqslant y^*$. Suppose, that $z^*=y^*$, then according to Lemma 1.2 we have $\bar{z}y\in I$. Now $\bar{z}y=(\bar{p}\bar{q}x)^- y=py+qy+\bar{x}y$. Now $q\leqslant\bar{p}x=(\bar{y}x)^- x=yx$, thus $q\leqslant yx$

and thus $q \leqslant y$ and $qy = q$. $py = \bar{y}xy = 0$. Thus $\bar{z}y = q + \bar{x}y \in I$ and thus $q \in I$ but $q \in A - I$. Therefore $z^* < y^*$. (2) We must show that $\bar{z}x_\gamma \in I$. Now $\bar{z}x_\gamma = (\bar{p}\bar{q}x)^- x_\gamma = (p + q + \bar{x})x_\gamma = px_\gamma + qx_\gamma + \bar{x}x$. We have

$$\bar{x}x_\gamma = 0, \; px_\gamma = p_\gamma \in I, \; qx_\gamma = q_\gamma \in I,$$

thus $\bar{z}x_\gamma \in I$. It follows from (1) and (2) that $\sum_{\gamma \in \Gamma} x_\gamma^*$ does not exist.

In the following theorem we shall show that to every infinite disjointed set of ideals of a complete Boolean algebra corresponds a quotient algebra which is not complete.

Theorem 4.2

If A is a complete Boolean algebra and $\{I_\gamma, \gamma \in \Gamma\}$ is an infinite set of disjoint ideals (the set-theoretic intersection of any two distinct ideals is the zero element), $I_\gamma \neq (0)$ for every $\gamma \in \Gamma$, and I is the ideal generated by the ideals I_γ, then A/I is not complete.

Proof.

It will be convenient to denote the ideals of the given set by

$$\{I_{\gamma,\delta}, \; \gamma \in \Gamma, \; \delta \in \Gamma\}$$

which is of course always possible since the set is infinite. Since every $I_{\gamma,\delta} \neq (0)$ we can choose in every $I_{\gamma,\delta}$ an element $x_{\gamma,\delta} \neq 0$. Now let for every $\gamma \in \Gamma$, $x_\gamma = \sum_{\delta \in \Gamma} x_{\gamma,\delta}$ and let $x = \sum_{\gamma \in \Gamma} x_\gamma$. We shall prove that $\sum_{\gamma \in \Gamma} x_\gamma^*$ does not exist. First we note that for every $\gamma \in \Gamma$, $x_\gamma < 1$. Indeed suppose, that for some $\gamma' \in \Gamma$, $x_{\gamma'} = 1$. Let $\gamma'' \in \Gamma$ be an index such that $\gamma'' \neq \gamma'$. Then we have

$$x_{\gamma''} = 1x_{\gamma''} = x_{\gamma'}x_{\gamma''} = \sum_{\delta \in \Gamma} x_{\gamma',\delta} \sum_{\delta \in \Gamma} x_{\gamma'',\delta} = 0,$$

applying the infinite distributive law (3) of § 1 and the fact that the ideals of the set are disjoint. But if $x_{\gamma''} = 0$, then $x_{\gamma'',\delta} = 0$ for all $\delta \in \Gamma$ contradictory to $x_{\gamma'',\delta} \neq 0$ for all $\delta \in \Gamma$. Now suppose p is some element of A, such that $p \leqslant x$ and $p_\gamma = px_\gamma \in I$ for every $\gamma \in \Gamma$. Then $\sum_{\gamma \in \Gamma} p_\gamma = px = p$. Now let γ' be some element of Γ, then we have that $p_{\gamma'}$ is the sum of a finite number of elements each of which belongs to some $I_{\gamma,\delta}$, thus

$$p_{\gamma'} = \sum_{i=1}^{i=n} y_{\gamma_i,\delta_i}, \; y_{\gamma_i,\delta_i} \in I_{\gamma_i,\delta_i}, \; \gamma_i \in \Gamma, \delta_i \in \Gamma, \; i = 1, 2, \ldots, n.$$

Now we assert that for every i for which $\gamma_i \neq \gamma'$, we have $\gamma_{\gamma_i,\delta_i} = 0$. Indeed suppose that for some $i = j$, $1 \leqslant j \leqslant n$, we have $\gamma_j \neq \gamma'$, then

$$p_{\gamma'} = px_{\gamma'} = p \sum_{\delta \in \Gamma} x_{\gamma',\delta} = \sum_{\substack{i=1 \\ i \neq \gamma}}^{i=n} y_{\gamma_i,\delta_i} + y_{\gamma_j,\delta_j}$$

and multiplying either side with y_{γ_j,δ_j}, we get $0 = y_{\gamma_j,\delta_j}$, since the ideals are disjoint. Thus it follows that every p_γ can be written as

$$p_\gamma = \sum_{i=1}^{i=n(\gamma)} y_{\gamma,\delta_i},\ y_{\gamma,\delta_i} \in I_{\gamma,\delta_i},\ \gamma \in \Gamma, \delta_i \in \Gamma,\ i = 1, 2, \ldots, n(\gamma),$$

Now since Γ is infinite, we can choose for every $\gamma \in \Gamma$ an element $x_{\gamma,\delta(\gamma)}$ such that $\delta(\gamma) \neq \delta_i$ for $i=1, 2, \ldots, n(\gamma)$. (We note that $x_{\gamma,\delta(\gamma)} \in I$ for every $\gamma \in \Gamma$.) Now let $q = \sum_{\gamma \in \Gamma} x_{\gamma,\delta(\gamma)}$ and let for every $\gamma \in \Gamma$, $q_\gamma = qx_\gamma$. We shall show that:

(1) $q_\gamma \in I$ for every $\gamma \in \Gamma$, (2) $q \leqslant \bar{p}x$, (3) $q \in A - I$.

(1) We have for every

$$\gamma \in \Gamma, q_\gamma = qx_\gamma = (\sum_{\lambda \in \Gamma} x_{\lambda,\delta(\lambda)})x_\gamma = \sum_{\lambda \in \Gamma} x_{\lambda,\delta(\lambda)} \sum_{\sigma \in \Gamma} x_{\gamma,\sigma} = x_{\gamma,\delta(\gamma)}$$

since the set $\{x_{\gamma,\delta}\}$ is disjointed. Now $x_{\gamma,\delta(\gamma)} \in I$ thus $q_\gamma \in I$.

(2) We have for every

$$\gamma \in \Gamma, p_\gamma x_{\gamma,\delta(\gamma)} = (\sum_{i=1}^{i=n(j)} y_{\gamma,\delta_i})x_{\gamma,\delta(\gamma)} = \sum_{i=1}^{i=n(j)} y_{\gamma,\delta_i} x_{\gamma,\delta(\gamma)} = 0$$

since $\delta(\gamma) \neq \delta_i$ for $i=1, 2, \ldots, n(\gamma)$. Furthermore we have for every γ and

$$\gamma' \in \Gamma, \gamma \neq \gamma', p_{\gamma'} x_{\gamma,\delta(\gamma)} = (\sum_{i=1}^{i=n(\gamma')} y_{\gamma',\delta_i})x_{\gamma,\delta(\gamma)} = \sum_{i=1}^{i=n(\gamma')} y_{\gamma',\delta_i} x_{\gamma,\delta(\gamma)} = 0.$$

Therefore it follows that $p_\lambda x_{\gamma,\delta(\gamma)} = 0$ for all $\lambda, \gamma \in \Gamma$. Thus also

$$(\sum_{\lambda \in \Gamma} p_\lambda) x_{\gamma,\delta(\gamma)} = px_{\gamma,\delta(\gamma)} = 0.$$

Now according to (1) we have $q_\gamma = x_{\gamma,\delta(\gamma)}$ and thus $pq_\gamma = 0$ or $q_\gamma \leqslant \bar{p}$ (according to Lemma 1.2) for every $\gamma \in \Gamma$. We also have that $q_\gamma \leqslant x$ and thus $q_\gamma \leqslant \bar{p}x$ and thus $q \leqslant \bar{p}x$.

(3) $q = \sum_{\gamma \in \Gamma} x_{\gamma,\delta(\gamma)}$. Suppose $q \in I$, then q can be written as

$$q = \sum_{i=1}^{i=n} z_{\gamma_i,\delta_i},\ z_{\gamma_i,\delta_i} \in I_{\gamma_i,\delta_i}, \gamma_i, \delta_i \in \Gamma,\ i = 1, 2, \ldots, n.$$

Therefore we have

$$\sum_{\gamma \in \Gamma} x_{\gamma,\delta(\gamma)} = \sum_{i=1}^{i=n} z_{\gamma_i,\delta_i}.$$

Now since Γ is infinite there exists a $\gamma' \in \Gamma$, such that $\gamma' \neq \gamma_i$ for $i=1, 2, \ldots, n$. Then we have

$$x_{\gamma',\delta(\gamma')} \sum_{\gamma \in \Gamma} x_{\gamma,\delta(\gamma)} = x_{\gamma',\delta(\gamma')} = \sum_{i=1}^{i=n} x_{\gamma',\delta(\gamma')} z_{\gamma_i,\delta_i} = 0,$$

since the ideals $I_{\gamma,\delta}$ are disjoint. However $x_{\gamma',\delta(\gamma')} \neq 0$ and thus it follows that $q \in A - I$. It follows from (1), (2) and (3) that q and the elements q_γ satisfy the conditions of Theorem 4.1 and therefore we may conclude that $\sum_{\gamma \in \Gamma} x_\gamma^*$ does not exist. This completes the proof of the theorem.

We shall apply Theorem 4.2 to the following example. Let W be an infinite set of elements and A the complete Boolean algebra of all of its subsets. Let $\{P_\gamma, \gamma \in \Gamma\}$ be an infinite set of disjoint non-void subsets of W. Let I be the ideal of A which consists of all subsets of W which are the set-theoretic union of a finite number of subsets of W, each of which is a subset of some P_γ. Then A/I is not complete. It follows in particular that the Boolean algebra of all the subsets of an infinite set W modulo finite subsets of W is not complete. We shall now state the main theorem of this section.

Theorem 4.3

Every infinite complete Boolean algebra A has a quotient algebra which is not complete.

Proof.

According to Theorem 2.2, A has a countably infinite disjointed subset of elements $\{x_i, i=1, 2, \ldots\}$, $x_i x_j = 0$ for $i \neq j$. Let I be the ideal generated by the set of principal ideals $\{(x_i), i=1, 2, \ldots\}$ then it follows from Theorem 4.2 that the quotient algebra A/I is not complete.

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REFERENCES

1. Birkhoff, Lattice Theory. Revised edition. New York (1948).
2. Stone, M. H., The theory of representations for Boolean algebras, Transac. of the Amer. Math. Soc. **40**, 37–111 (1936).
3. Smith, Jr., E. C. and A. Tarski, Higher degrees of distributivity and completeness in Boolean algebras, Transac. of the Amer. Math. Soc. **84**, 230–257 (1957).